Multinerves and Helly Numbers of Acyclic Families

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ABSTRACT

The nerve of a family of sets is a simplicial complex that records the intersection pattern of its subfamilies. Nerves are widely used in computational geometry and topology, because the nerve theorem guarantees that the nerve of a family of geometric objects has the same topology as the union of the objects, if they form a good cover.

In this paper, we relax the good cover assumption to the case where each subfamily intersects in a disjoint union of possibly several homology cells, and we prove a generalization of the nerve theorem in this framework, using spectral sequences from algebraic topology. We then deduce a new topological Helly-type theorem that unifies previous results of Amenta, Kalai and Meshulam, and Matoušek. This Helly-type theorem is used to (re)prove, in a unified way, bounds on transversal Helly numbers in geometric transversal theory.

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General Terms: Theory

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1. INTRODUCTION

The *nerve* of a family F of sets is a combinatorial structure that encodes the intersection patterns of subfamilies of F. More precisely, it is an abstract simplicial complex with vertex set F where $G \subseteq F$ forms a simplex if and only if the intersection of the subfamily G is non-empty.

The notion of nerve is widely used in computational geometry (see discussion below), primarily because, under adequate assumptions, it provides a faithful combinatorial description of the topology of the union of the sets. Indeed, if the family is geometric, the *nerve theorem*, due to Borsuk [8], states that the nerve is homotopy equivalent to the union of the sets in the family, provided that the family is a *good cover*: the intersection of any subfamily is either empty or contractible.

In the nerve theorem, the requirement that the family be a good cover is rather strong and not always adequate. Here, we consider *acyclic* families of sets, where every subfamily intersects in finitely many connected components, each of which is a homology cell (in particular, a set that is contractible is a homology cell). The contributions of our paper are threefold:

1. We define an analogue of the nerve, called the *multinerve*, that is suitable for general acyclic families, and we prove that this combinatorial structure enjoys an analogue of the nerve theorem.

2. Furthermore, we demonstrate the usefulness of this construction by proving a new *topological Helly-type theorem* for acyclic families. Helly-type theorems are results of the following form: Given a family F of sets, if every subfamily of cardinality at most h has non-empty intersection, then the whole family has non-empty intersection. In such a case, F is said to have *Helly number* at most h. Our theorem generalizes previous results of Amenta [3], Kalai and Meshulam [33], and Matoušek [36].

3. Finally, this result is used to (re)prove, in a unified way, bounds on transversal Helly numbers in *geometric transversal theory*.

Before describing our results in more detail, we review various incarnations of nerves in discrete and computational geometry.

Nerves in discrete and computational geometry.

While the study of unions and intersections of basic shapes is a classical topic in discrete and computational geometry, the methods used to study these structures have recently evolved toward a greater integration of geometric, combinatorial, and topological arguments. Several of them can be traced back to the classical notion of nerve:

- Standard objects relate to or are particular cases of nerve complexes: (a) the *intersection graph* of a family of sets is the 1-dimensional skeleton of the nerve of that family, (b) the nerve of a family of geometric objects is the smallest abstract simplicial complex containing their *dual range space* (seen as a hypergraph),

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and (c) given a point set P and a parameter $t > 0$, the *Čech complex* $C(P,t)$ used in computational topology [11, 17, 6] is no other than the nerve of the family of balls of radius t centered in the points of P. In the latter case, the nerve theorem justifies that the Čech complex is a topologically faithful construction. Also the nerve theorem ensures that the restricted Delaunay triangulation of an object has the same topology as that object, under mild assumptions [21].

- Helly's theorem [29] and its topological generalization [30] have implications in combinatorial optimization, e.g., they allow to bound the combinatorial dimension of some LP-type problems [2], or in reconstruction, e.g., to define the maximum cardinality of a *blocker* [5]. Helly's topological theorem states that the Helly number of a good cover in $\mathbb{R}^d$ is at most $d+1$ (see also Debrunner [18]). For open sets, this follows easily from the nerve theorem. Indeed, if F is an open good cover in $\mathbb{R}^d$ such that[1] $\bigcap_F = \emptyset$ and $\bigcap_G \neq \emptyset$ for all proper subfamilies $G \subsetneq F$, then $\mathcal{N}(F)$ is the boundary of a simplex with $|F|$ vertices; hence, $\mathcal{N}(F)$ has non-zero homology in dimension $|F|-2$. On the other hand, $\mathcal{N}(F)$ has the same homology as $\bigcup_F$, which, as an open set in $\mathbb{R}^d$, has zero homology in dimension d and larger. So $|F|-2$ is at most $d-1$. It follows that if all subsets of F of size $d+1$ have non-empty intersection then $\bigcap_F$ is non-empty.

- The inclusion-exclusion formula asserts that for any family F of subsets of $\mathbb{R}^d$ we have

$$\mathbf{1}_{\bigcup_F} = \sum_{\emptyset \subsetneq G \subseteq F} (-1)^{|G|+1} \mathbf{1}_{\bigcap_G},$$

  where $\mathbf{1}_X$ denotes the indicator function of set X. This allows, for instance, to evaluate integral quantities such as volumes for $\bigcup_F$ (a fundamental question in bioinformatics) from the values of that quantity for $\bigcap_G$, where $G \subseteq F$. This identity has a topological flavor to it: for any point $p \in \mathbb{R}^d$, if F_p denotes the subset of F of sets that contain p, then

$$\sum_{\emptyset \subsetneq G \subseteq F} (-1)^{|G|+1} \mathbf{1}_{\bigcap_G}(p) = \sum_{\emptyset \subsetneq G \subseteq F_p} (-1)^{|G|+1}$$

  which equals zero if $p \notin \bigcup_F$ and the Euler characteristic of the simplex with $|F_p|$ vertices, namely one, if $p \in \bigcup_F$. This point of view allows to characterize and compute, for certain families of objects, more compact inclusion-exclusion formulas using tools such as the nerve theorem, as shown by Attali and Edelsbrunner [4].

Previous work on Helly numbers.

Previous bounds on Helly numbers of families of non-connected sets come in two different flavors. On the one hand, one can start with a "ground" family $\mathcal{H}$ whose Helly number is bounded and consider families F such that the intersection of any subfamily $G \subseteq F$ is a disjoint union of at most r elements of $\mathcal{H}$. When $\mathcal{H}$ is closed under intersection and *non-additive* (that is, the union of two disjoint elements of $\mathcal{H}$ is never an element of $\mathcal{H}$) the Helly number of F can be bounded by r times the Helly number of $\mathcal{H}$. This was conjectured (and proven for $r = 2$) by Grünbaum and Motzkin [27] and a proof of the general case was recently published by Eckhoff and Nischke [20], building on ideas of Morris [38]. Direct proofs were also given by Amenta [3] in the case where $\mathcal{H}$ is a finite family of compact convex sets in $\mathbb{R}^d$ and by Kalai and Meshulam [33] in the case where $\mathcal{H}$ is a good cover in $\mathbb{R}^d$. On the other hand, Matoušek [36] and Alon and Kalai [1] showed, independently, that if F is a family of sets in $\mathbb{R}^d$ such that the intersection of any subfamily is the union of at most r (possibly intersecting) convex sets, then the Helly number of F can be bounded from above by some function of r and d. Matoušek also gave a topological analogue [36, Theorem 2]: he bounds from above (again, by a function of r and d) the Helly number of families of sets in $\mathbb{R}^d$ assuming that the intersection of any subfamily has at most r connected components, each of which is $(\lceil d/2 \rceil - 1)$-connected, that is, has its ith homotopy group vanishing for $i \leq \lceil d/2 \rceil - 1$.

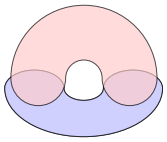

Figure 1. A family of two sets that is not a good cover but satisfies the hypotheses of our theorem.

Our results in more detail.

While powerful generalizations of nerves were developped in algebraic topology, their statements require heavy formalism with elaborated algebraic structures (spectral sequences, in particular), which prove difficult to manipulate for the non-expert. The first contribution of our paper is the definition of an analogue of the nerve, called the *multinerve*, and a generalization of the nerve theorem to the multinerve, that is relatively easy to state and can be used as a black box in applications. The multinerve is not a simplicial complex, but a more general *simplicial poset*. The proof of our "multinerve theorem" is based on Leray's acyclic cover theorem (see [9, 34, 10, 23, 22] for instance) and Čech complexes from algebraic topology.[2]

Our second contribution is a topological Helly-type theorem. We prove that if F is a family of open sets in $\mathbb{R}^d$ such that every subfamily intersects in at most r connected components, each of which is a homology cell, then the Helly number of F is at most $r(d+1)$. We obtain this Helly-type theorem by revisiting the proof of Kalai and Meshulam and replacing a certain construction by the multinerve, demonstrating the effectiveness of this new tool. This generalizes previous results of Amenta [3] and Kalai and Meshulam [33]: Figure 1 shows a family for which our result applies with $r = 2$, but where the results of Amenta and of Kalai and Meshulam do not (as the family of connected components is not a good cover). Matoušek proved [36] that the Helly numbers of the families we consider were bounded; we provide a tight, explicit bound. Finally, our result and the Eckhoff-Morris-Nischke theorem [20] do not seem to imply

[1] Here and in the rest of the paper we use $\bigcap_A$ and $\bigcup_A$ to denote, respectively, $\bigcap_{x \in A} x$ and $\bigcup_{x \in A} x$.

[2] Let us emphasize that here we mean a *Čech complex* in the sense of algebraic toplogy, which is a more general object than what the eponymous structure generally used in computational topology.

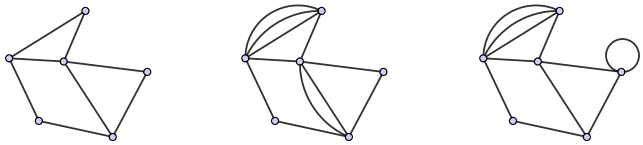

Figure 2. One-dimensional simplicial complex (left) vs simplicial poset (middle). Simplicial sets (right) are even more general structures.

one another, but to be distinct generalizations of the Kalai-Meshulam theorem.

Also, an interesting feature of our result is that it generalizes to families living in rather general topological spaces, not only $\mathbb{R}^d$. Our bound on Helly numbers, when applied in a k-dimensional submanifold embedded in $\mathbb{R}^d$, depends only on k and is oblivious to d. Since Helly numbers arise in the context of surface reconstruction as the maximum size of blockers [5], this could be a useful tool to overcome the "curse of dimensionality", a major issue in high-dimensional manifold reconstruction and related areas.

As a last contribution, we use our new Helly-type theorem to reprove or sharpen, in a unified way, bounds on Helly numbers in *geometric transversal theory*. While most bounds were previously known, they were obtained through ad hoc, and sometimes tedious, methods. In essence, it turns out that the set of lines piercing a given family of objects in $\mathbb{R}^d$ is, in many cases, a union of disjoint contractible sets in line space; our Helly-type result is thus suitable for bounding the Helly numbers of such sets of transversals.

Due to space limitations, we restrict the presentation in this conference version to the main ideas and proof outlines, and refer to the complete version [15] for full details.

2. MULTINERVES

In this section we recall the definitions of simplicial complexes and simplicial posets and introduce the multinerve, a simplicial poset that refines the notion of nerve.

2.1 Simplicial Complexes and Simplicial Posets

At the combinatorial level, an (abstract) *simplicial complex* X over a set of *vertices* V is a non-empty family of subsets of V closed under taking subsets; in particular, $\emptyset$ belongs to every simplicial complex. An element σ of X is a *simplex*; its *dimension* is the cardinality of σ minus one; a d-simplex is a simplex of dimension d.

Intuitively, a simplicial partially ordered set (simplicial poset for short) is a set of simplices with an incidence relation; a d-simplex still has $d+1$ distinct vertices; however, in contrast to simplicial complexes, there may be several simplices with the same vertex set, but no two can be incident to the same higher-dimensional simplex (see Figure 2).

Formally, let X be a finite set and $\preceq$ a partial order on X; we also say that $(X, \preceq)$ is a partially ordered set, or that X is a *poset* to save breath. Let $[\alpha, \beta] = \{\tau \in X \mid \alpha \preceq \tau \preceq \beta\}$ denote the *segment* defined by α and β. (Similarly, we will denote by $(\alpha, \cdot]$ the set $\{\tau \in X \mid \alpha \prec \tau\}$, and so on.) A map $\varphi : X \to Y$ between two posets $(X, \preceq^X)$ and $(Y, \preceq^Y)$ is *monotone* if it preserves the order: for any $\sigma, \tau \in X$ $\sigma \preceq^X \tau \Rightarrow \varphi(\sigma) \preceq^Y \varphi(\tau)$. An *isomorphism* of posets is a monotone bijection between them. For an overview of basic properties of simplicial posets see, e.g., Stanley [42].

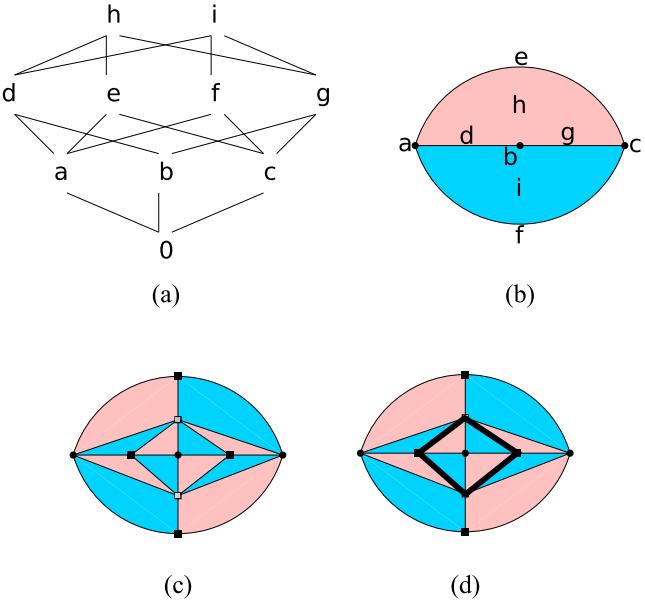

Figure 3. (a) A simplicial poset X. (b) The geometric realization of X. (c) The geometric realization of $\mathrm{sd}(X)$, which also equals $\dot{D}_X(0)$. (d) $\dot{D}_X(b)$ is a 1-dimensional simplicial complex that is a cycle of length four (in black bold lines).

A poset X is a *simplicial poset* if it satisfies two conditions (see Figure 3(a)). First, X must have a *least element* 0, that is $0 \preceq \sigma$ for any $\sigma \in X$. Second, for any $\sigma \in X$, there must exist some integer d such that the lower segment $[0, \sigma]$ is isomorphic to $2^{\{0,\dots,d\}}$, the poset of faces of a d-simplex partially ordered by the inclusion; d is then called the *dimension* of σ. The elements of X are called its *simplices* and the simplices of dimension 0 (i.e. that only dominate 0) are its *vertices*. If $\tau \preceq \sigma$ we also say that τ is *contained in* (or *a face of*) σ. Simplicial posets lie in-between simplicial complexes and the more general notion of *simplicial sets* used in algebraic topology [37, 24]. The simplices of a simplicial complex, ordered by inclusion, form a simplicial poset (with $\emptyset$ as least element). The converse is not always true: the one-dimensional simplicial complexes are precisely the graphs without loops or multiple edges, while the one-dimensional simplicial posets correspond to the graphs without loops but possibly with multiple edges (refer back to Figure 2). If τ is a simplex of a simplicial poset with set of vertices V, the map that associates to any face of τ the set of vertices of that face is a bijection between $[0, \tau]$ and 2^V. There may, however, exist several simplices with the same set of vertices, but no two of them can be faces of one and the same simplex. Just as for simplicial complexes, every simplicial poset has an associated topological space, its *geometric realization*; see Figure 3(b).

2.2 Multinerves: Definition and Properties

Let F be a finite family of subsets of a topological space. We define the *multinerve* $\mathcal{M}(F)$ of F as:

$$\mathcal{M}(F) = \left\{ (C, A) \;\middle|\; \begin{array}{l} A \subseteq F, \\ C \text{ is a connected component of } \bigcap_A \end{array} \right\}.$$

By convention, we put $\bigcap_\emptyset = \bigcup_F$, and in particular, $(\bigcup_F, \emptyset)$ belongs to $\mathcal{M}(F)$. We turn $\mathcal{M}(F)$ into a poset by equipping

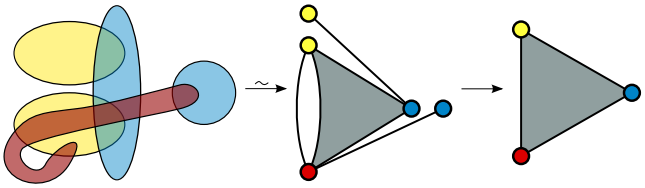

Figure 4. Left: A family F of subsets of $\mathbb{R}^2$. Middle: Its multinerve $\mathcal{M}(F)$. Right: Its nerve $\mathcal{N}(F)$.

it with the partial order:

$$(C', A') \preceq (C, A) \Leftrightarrow C \supseteq C' \text{ and } A \subseteq A'.$$

To get an intuition, it does not harm to assume that, whenever A and A' are different subsets of F, the connected components of $\bigcap_A$ and of $\bigcap_{A'}$ are different. Under this assumption, $\mathcal{M}(F)$ can be identified with the set of all connected components of the intersection of any subfamily of F, equipped with the opposite of the inclusion order. See Figure 4 for an example.

LEMMA 1. *The poset $\mathcal{M}(F)$ is simplicial.*

PROOF. The projection on the second coordinate identifies any lower segment $[(\bigcup_F, \emptyset), (C, A)]$ with the simplex 2^A. Indeed, let $A' \subseteq A$ and let $C' \subseteq \bigcup_F$. The lower segment $[(\bigcup_F, \emptyset), (C, A)]$ contains (C', A') if and only if C' is the connected component of $\bigcap_{A'}$ containing C. Moreover, by definition, $\mathcal{M}(F)$ contains a least element, namely $(\bigcup_F, \emptyset)$. The statement follows. □

Intuitively, $\mathcal{M}(F)$ is an "expanded" version of $\mathcal{N}(F)$: while $\mathcal{N}(F)$ has one simplex for each non-empty intersecting subfamily, $\mathcal{M}(F)$ has one simplex for each *connected component* of an intersecting subfamily.

3. MULTINERVE THEOREM

In this section, we state a "homological multinerve theorem" (Theorem 3): under certain conditions on a family F, the multinerve $\mathcal{M}(F)$ captures the homology of $\bigcup_F$. Before, we explain how the notion of homology for simplicial complexes extends, in the natural way, to simplicial posets.

3.1 Homology of Simplicial Posets

The *homology* of a simplicial poset can be defined as a direct extension of simplicial homology for simplicial complexes (or, alternatively, as the singular homology of the geometric realization of the simplicial poset, but we deliberately avoid this point of view here). Let X be a simplicial poset and assume chosen an ordering on the set of vertices of X. For $n \geq 0$, let $C_n(X)$ be the $\mathbb{Q}$-vector space with basis the set of simplices of X of dimension exactly n. If σ is an n-dimensional simplex, the lower segment $[0, \sigma]$ is isomorphic to the poset of faces of a standard n-simplex $2^{\{0,\dots,n\}}$; here we choose the isomorphism so that it preserves the ordering on the vertices. Thus, we get $n+1$ faces $d_i(\sigma) \in X$ (for $i = 0, \dots, n$), each of dimension $n-1$: namely, $d_i(\sigma)$ is the (unique) face of σ whose vertex set is mapped to $\{0, \dots, n\} \setminus \{i\}$ by the above isomorphism. Extending the maps d_i by linearity, we get the *face operators* $d_i : C_n(X) \to C_{n-1}(X)$. Let $d = \sum_{i=0}^{n} (-1)^i d_i$ be the linear map $C_n(X) \to C_{n-1}(X)$ (which is defined for any $n \geq 0$). The fact that $d \circ d = 0$ is easy and follows from the same argument as for simplicial complexes since it is computed inside the vector space generated by $[0, \sigma]$ which is isomorphic to a standard simplex. The (simplicial) nth homology group $H_n(C_\bullet(X), d)$ is defined as the quotient vector space of the kernel of $d : C_n(X) \to C_{n-1}(X)$ by the image of $d : C_{n+1}(X) \to C_n(X)$. The simplicial homology of a simplicial poset coincides with the homology of its geometric realization.

We emphasize that, in this paper, we only consider homology over $\mathbb{Q}$. In the sequel, we denote by $H_i(O)$ the ith $\mathbb{Q}$-homology group of O (whether O is a simplicial poset or a topological space), and by $\tilde{H}_i(O)$ the corresponding *reduced* homology group (see, e.g., Hatcher [28]). A homology cell is a set O such that $\tilde{H}_i(O) = 0$ for each i; in particular, every contractible space is a homology cell.

3.2 Multinerve Theorem

Let Γ be a locally arc-wise connected topological space. If F is a good cover of Γ then the multinerve of F is simply its nerve, and the nerve theorem asserts that $\bigcup_F$ and $\mathcal{M}(F)$ have the same homotopy type. We prove a homological version of the nerve theorem that is valid for multinerves of *acyclic families*, where a family is *acyclic* if for any non-empty subfamily $G \subseteq F$, each connected component of the intersection of the elements of G is a $\mathbb{Q}$-homology cell.

THEOREM 2. *Let F be a family of open sets in a locally arc-wise connected topological space Γ. If F is* acyclic*, then $\tilde{H}_\ell(\mathcal{M}(F)) \cong \tilde{H}_\ell(\bigcup_F)$ for any non-negative integer ℓ.*

Any good cover is acyclic, as a contractible space is a homology cell, but the reverse is not true as (a) acyclicity allows a subfamily to intersect in more than one connected component, and (b) there exist homology cells that are not contractible. See Figures 1 and 4 for examples of acyclic families that are not good covers. In fact, we slightly strengthen Theorem 2 by allowing intersections of few elements of the family to have non-zero homology in low dimension; this variant will be useful for our applications to geometric transversal theory. Specifically, we call a family F of subsets of Γ *acyclic with slack s* if for every non-empty subfamily $G \subseteq F$ and every $i \geq \max(1, s - |G|)$ we have $\tilde{H}_i(\bigcap_G) = 0$. Here is the strengthened version of Theorem 2.

THEOREM 3 (HOMOLOGICAL MULTINERVE THEOREM). *Let F be a family of open sets in a locally arc-wise connected topological space Γ. If F is* acyclic with slack s *then $\tilde{H}_\ell(\mathcal{M}(F)) \cong \tilde{H}_\ell(\bigcup_F)$ for any non-negative integer $\ell \geq s$.*

We only provide a sketch of proof, which is independent from the rest of the paper and can safely be skipped.

SKETCH OF PROOF OF THEOREMS 2 AND 3. The homology of topological spaces can be computed as the homology of standard chain complexes [28]. (Recall that a chain complex is a graded vector space $E_\bullet = \oplus_n E_n$ and a differential, that is, a linear map $d : E_n \to E_{n-1}$ such that $d \circ d = 0$, and that its homology groups $H_n(C_\bullet(X), d)$ are the quotients of the kernels of $d : E_n \to E_{n-1}$ by the images of $d : E_{n+1} \to E_n$.)

The key idea of the proof is to compute the homology of the union $\bigcup_F$ using a peculiar chain complex, the *Čech complex* $\check{C}_\bullet(F)$, that depends not only on $\bigcup_F$ but on the

way the family F covers that union. Intuitively, $\check{C}_\bullet(F)$ takes into account the nerve of F as well as the topology of the intersections of subfamilies. Roughly, the vector space of $\check{C}_\bullet(F)$ is

$$\check{C}_\bullet(F) = \bigoplus_n \Big(\bigoplus_{p+|G|=n+1} C_p(\bigcap\nolimits_G) \Big)$$

where $C_p(X))$ is the degree-p subspace of a standard[3] chain complex computing the homology of the topological space X. The differential $d : \check{C}_\bullet(F) \to \check{C}_{\bullet-1}(F)$ is obtained from the differentials of the chain complexes $C_\bullet(\bigcap_G)$ using an analogue of the inclusion-exclusion principle. We refer to the full version for a formal definition, or to standard textbooks [10, 23] (or [9] for an analogue in de Rham cohomology).

A key feature of the Čech complex is that its homology (and thus the one of the topological space $\bigcup_F$) can be computed by *successive approximations*, starting by the direct sum of the homology of the topological spaces $\bigcap_G$ for all subfamilies; then the differential on the Čech complex will induce another differential (of combinatorial nature) on this direct sum whose homology groups will be a new approximation of the homology of the Čech complex (and thus of the space $\bigcup_F$). This new approximation also inherits a differential so that we can compute a third approximation, and so on. These approximation eventually yields the homology of the Čech complex.

This idea is made mathematically rigorous by *spectral sequences*. A spectral sequence is a sequence $(E^r_{\bullet,\bullet})_{r\geq 0}$ of bigraded vector spaces $E^r_{p,q}$ $(p, q \geq 0)$ equipped with a differential $d^r : E^r_{p,q} \to E^r_{p-r,q+r-1}$ such that $E^{r+1}_{\bullet,\bullet} \cong H_\bullet(E^r_{\bullet,\bullet}, d^r)$ is the homology of $E^r_{\bullet,\bullet}$ with respect to the differential d^r. The term $E^r_{\bullet,\bullet}$ is called the r-page of the spectral sequence. See Figure 5 for an example. Note that, for degree reasons, $d^r : E^r_{p,q} \to E^r_{p-r,q+r-1} = 0$ for $r > p$. It follows that the terms $E^r_{p,q}$ *stabilize*, that is, $E^r_{p,q} \cong E^{r+1}_{p,q} \cong \cdots$ for r large enough. We write $E^\infty_{p,q}$ for the stabilized groups $E^\infty_{p,q} \cong \cdots \cong E^{p+2}_{p,q} \cong E^{p+1}_{p,q}$. The homology of the Čech complex can be read off from the stabilized groups; precisely there is a linear isomorphism of vector spaces $H_n(\check{C}(F)) \cong \bigoplus_{p+q=n} E^\infty_{p,q}$ for every $n \geq 0$. Let us note that the fact that the Čech complex can be computed by such a spectral sequence follows from the observation that it is endowed with a natural *filtration* given by the cardinality of the subfamilies $G \subseteq F$; namely, for $q \geq 0$, we have vector spaces $\big(\check{C}_\bullet(F)\big)^p$ obtained from $\check{C}_\bullet(F)$ by considering only families G such that $|G| \leq p + 1$. Then the page E^0 of the spectral sequence is just $E^0_{p,q} := \big(\check{C}_{p+q}(F)\big)^p / \big(\check{C}_{p+q}(F)\big)^{p-1}$ equipped with the differential induced by d on the quotient spaces. When the family consists of only two open sets, our computation of the homology of a space using the above spectral sequence boils down to the standard Mayer-Vietoris exact sequence [28, 41].

A *key property* of spectral sequences is the following result: assume that there exists integers N and $r \geq 0$ such that

[3]To prove that the Čech complex has the same homology as the topological space $\bigcup_F$ given by the union, a technical point here is that one needs to use "nice enough" chain complexes computing homology, for instance as given by cosheaf theory. However this technical point can be harmlessly disregarded since it has no influence when computing the E^1 and onward pages of the spectral sequence below.

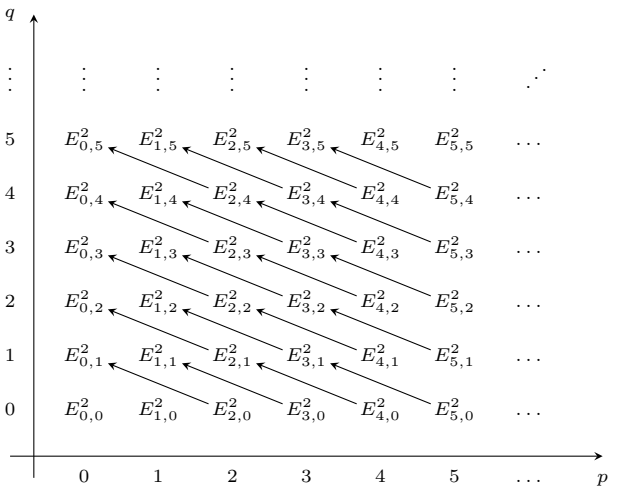

Figure 5. General form of the E^2-page of a spectral sequence. The arrows show the differential d^2.

$E^r_{p,q} = 0$ whenever $p + q \geq N$. Then $H_k(\check{C}_\bullet(F), d) = 0$ for $k \geq N$ as well.

We now explain what the spectral sequence for the Čech complex looks like with our assumptions on the family. First, the E^1-page is given by $E^1_{p,q} = \bigoplus_{|G|=p+1} H_q\big(\bigcap_G\big)$. The differential $d^1 : E^1_{p,q} \to E^1_{p-1,q}$ can be computed by formulae analogous to the one giving the differential of a simplicial poset. In fact, restricting to degree $q = 0$, $(E^1_{p,0}, d^1)$ is precisely $\big(C_p\big(\mathcal{M}(F)\big), d\big)$ and thus the component of the E^2-page in degree zero, $E^2_{p,0}$, is $H_p\big(\mathcal{M}(F)\big)$, the homology of the multinerve (viewed as a simplicial poset).

Thus for general covers, the E^2-page consists of bigraded vector spaces $E^2_{p,q}$ where $E^2_{p,0}$ is the homology of the multinerve $\mathcal{M}(F)$ associated to the cover.

In the case of an acyclic cover, all the homology groups $H_q(\bigcap_G)$ vanish for $q > 0$ and thus all the differentials d^r $(r \geq 2)$ as well. Hence $E^1_{p,q>0} = E^2_{p,q>0} = 0$ and the spectral sequence stabilizes immediately at page E^2. It follows that the homology of the Čech complex is the homology of the multinerve, which proves Theorem 2.

In the case of a cover that is acyclic with slack s, by assumption, for any $q \geq \max(1, s - p - 1)$ and any subfamily $G \subseteq F$ with $|G| = p + 1$, we have $H_q(\bigcap_G) = 0$ and thus $E^1_{p,q} = E^2_{p,q} = 0$ for $q \geq \max(1, s - p - 1)$ (see Figure 6). It follows that, for $p + q \geq s$, the spectral sequence stabilizes[4] at the page E^2, *i.e.*, $H_n(\check{C}(F)) \cong \bigoplus_{p+q=n} E^\infty_{p,q} = \bigoplus_{p+q=n} E^2_{p,q} = E^2_{n,0} = H_n(\mathcal{M}(F))$. Hence the homology of the Čech complex in degree greater or equal to s is the homology of the multinerve, which proves Theorem 3. □

4. HELLY NUMBERS OF ACYCLIC FAMILIES

The main result of this section is a Helly-type theorem for acyclic families.

[4]However, for $p + q < s$, it is in general necessary to compute higher order differential d^r to get some of the lower dimensional homology groups.

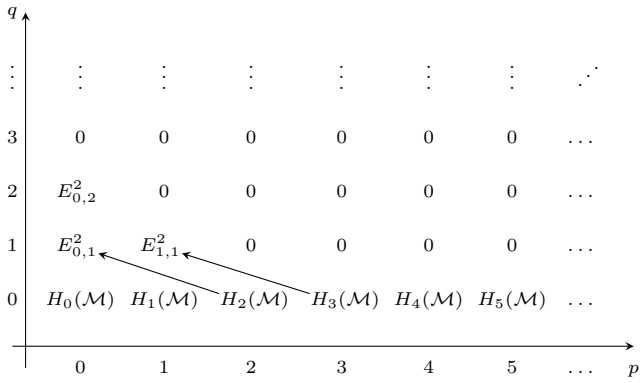

Figure 6. E^2-page of the Čech complex spectral sequence when F is acyclic with slack $s = 4$. The arrows show the only differential d^2 which can be non-zero. Here $H_i(\mathcal{M}(F))$ is abbreviated in $H_i(\mathcal{M})$.

4.1 Barycentric Subdivisions of Simplicial Posets

We first need some definitions related to the notion of barycentric subdivision of a simplicial poset [42]. Again, these definitions, when specialized to simplicial complexes, are the standard definitions from combinatorial topology; see the textbook of Matoušek [35, Chapter 1.7].

If $(Y, \preceq^Y)$ is a partially ordered set, a *chain* of Y is a subset of Y where any two elements are comparable with respect to $\preceq^Y$. The *order complex* of $(Y, \preceq^Y)$ is the set of all chains of Y ordered by inclusion; it is a simplicial complex.

Now, let X be a simplicial poset; we denote by 0 its least element. The *barycentric subdivision* of X, denoted by $\mathrm{sd}(X)$, is the order complex of $X \setminus \{0\}$. It turns out that (the geometric realizations of) X and $\mathrm{sd}(X)$ are homeomorphic; see Figure 3(c).

More generally, let σ be a simplex of X. We denote by $\dot{D}_X(\sigma)$ the order complex of $(\sigma, \cdot]$ in X. Thus, $\dot{D}_X(\sigma)$ is a subcomplex of the barycentric subdivision $\mathrm{sd}(X)$ of X (see Figure 3(d)), and coincides with $\mathrm{sd}(X)$ if $\sigma = 0$.

Finally, given $S \subseteq V$ of vertices, we let $X[S]$ denote the sub-poset of X *induced* by S, that is, the set of simplices of X whose set of vertices is contained in S.

4.2 Proof of Theorem 4

Recall that the *Helly number* of a family of sets with empty intersection is the size of its largest subfamily F such that (i) the intersection of all elements of F is empty, and (ii) for any proper subfamily $G \subsetneq F$, the intersection of the elements of G is non-empty.

Let Γ denote a locally arc-wise connected topological space. We let d_Γ denote the smallest integer such that every open subset of Γ has trivial $\mathbb{Q}$-homology in dimension d_Γ and higher; in particular, when Γ is a d-dimensional manifold, we have $d_\Gamma = d$ if Γ is non-compact or non-orientable and $d_\Gamma = d + 1$ otherwise (see, e.g., Greenberg [26, p. 121]); for example, $d_{\mathbb{R}^d} = d$. In this section, we prove the following Helly-type theorem.

THEOREM 4. *Let F be a finite family of open subsets of Γ. If F is acyclic and any subfamily of F intersects in at most r connected components, then the Helly number of F is at most $r(d_\Gamma + 1)$.*

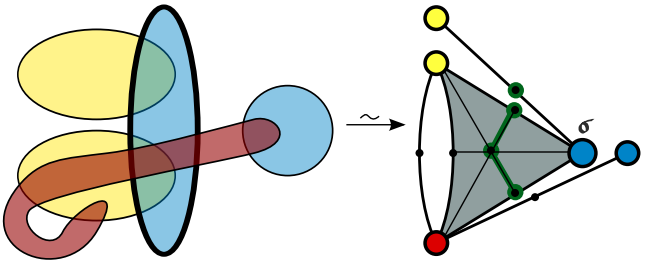

Figure 7. Continuation of Figure 4: On the left, the family F; on the right, the barycentric subdivision $\mathrm{sd}(\mathcal{M}(F))$ of the multinerve $\mathcal{M}(F)$. In this example, σ is a vertex of $\mathcal{M}(F)$ corresponding to one component C of an object in F. We see that $\dot{D}_{\mathcal{M}(F)}(\sigma)$ (in bold) is a subcomplex of $\mathrm{sd}(\mathcal{M}(F))$ that is the disjoint union of two homology cells. This is reflected in the fact that G_σ, the trace of the other objects of F inside C, is also the disjoint union of two homology cells.

We prove Theorem 4 in three steps. First, we associate to any simplicial poset X an index $J(X)$ that can be used to control Helly numbers. Specifically, given a simplicial poset X we let $J(X)$ be the smallest integer ℓ such that for every $j \geq \ell$, every $S \subseteq V$, and every simplex σ of $X[S]$, we have $\tilde{H}_j(\dot{D}_{X[S]}(\sigma)) = 0$.

LEMMA 5. *Any family F of sets has Helly number at most $J(\mathcal{N}(F)) + 1$.*

PROOF. Let $G \subseteq F$ be an inclusion-wise minimal subfamily with empty intersection. The nerve of G is the boundary of a $(|G| - 1)$-simplex and has therefore nontrivial homology in dimension $|G| - 2$. The nerve of G is the subcomplex of the nerve of F induced by G, that is $\mathcal{N}(G) = \mathcal{N}(F)[G]$, so we have that $\tilde{H}_{|G|-2}(\mathcal{N}(F)[G]) \neq 0$. Now, since $\dot{D}_{X[S]}(0)$ is the barycentric subdivision of $X[S]$, and therefore has the same homology groups as $X[S]$, it follows that $\tilde{H}_{|G|-2}(\dot{D}_{\mathcal{N}(F)[G]}(0)) \neq 0$ and $J(\mathcal{N}(F))$ is at least $|G| - 1$. The statement follows. □

As a side remark, we have only used the fact that for every $j \geq J(X)$ and every $S \subseteq V$, we have $\tilde{H}_j(X[S]) = 0$, which trivially follows from the definition of $J(X)$. The fact that J has a more subtle definition will be used later.

We next show that, under the assumptions of Theorem 4, the multinerve theorem yields an upper bound on $J(\mathcal{M}(F))$.

LEMMA 6. *If F is an acyclic family of open subsets of Γ then $J(\mathcal{M}(F)) \leq d_\Gamma$.*

PROOF. Let $G \subseteq F$ be a subfamily of F, and let σ be a simplex of $\mathcal{M}(F)[G] = \mathcal{M}(G)$. We need to prove that $\dot{D}_{\mathcal{M}(G)}(\sigma)$ has trivial reduced homology in dimension d_Γ and higher. The main idea is to show that $\dot{D}_{\mathcal{M}(G)}(\sigma)$ has the same homology as an open set in Γ. More precisely, let $\sigma = (C, A) \in \mathcal{M}(G)$. We define G_σ as the non-empty traces of the elements of $G \setminus A$ on C:

$$G_\sigma = \{U \cap C \mid U \in G \setminus A, U \cap C \neq \emptyset\}.$$

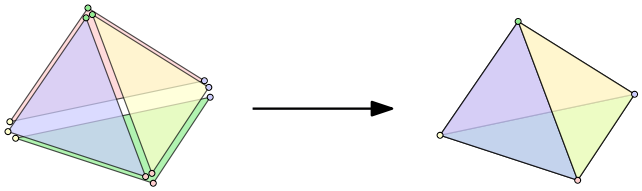

Figure 8. The projection of a disjoint union of 4 triangles (which has non-trivial homology only in dimension 0) onto the boundary of a simplex (which has non-trivial homology in dimension 2).

(Formally, G_σ is a multiset, as a given element may appear more than once.) We shall prove that $\dot{D}_{\mathcal{M}(G)}(\sigma)$ has the same homology as $\bigcup_{G_\sigma}$. See Figure 7.

Indeed, the map

$$\begin{cases} \mathcal{M}(G_\sigma) & \to & [\sigma,\cdot] \\ (C',A') & \mapsto & (C'\cap C, A\cup A') \end{cases}$$

is an isomorphism of posets. In particular, $[\sigma,\cdot]$ is a simplicial poset. Both posets have a least element, and removing them yields that $\mathcal{M}(G_\sigma)\backslash\{(\bigcup_G,\emptyset)\}$ and $(\sigma,\cdot]$ are isomorphic posets. Taking their order complexes, we get that $\dot{D}_{\mathcal{M}(G)}(\sigma)$ and $\mathrm{sd}(\mathcal{M}(G_\sigma))$ are isomorphic simplicial complexes.

Therefore, $\dot{D}_{\mathcal{M}(G)}(\sigma)$ has the same homology as $\mathcal{M}(G_\sigma)$. Since F is acyclic, the family G_σ is acyclic as well. Theorem 3 now ensures that the homology of $\mathcal{M}(G_\sigma)$ is the same as the homology of the union of the elements in G_σ. Since $\bigcup_{G_\sigma}$ is an open subset of Γ, it has homology zero in dimension d_Γ and higher. This concludes the proof. □

We now analyze what happens to the J-index when the simplicial poset $\mathcal{M}(F)$ is projected onto the simplicial complex $\mathcal{N}(F)$ by mapping each simplex of $\mathcal{M}(F)$ to its set of vertices:

$$\pi : \begin{cases} \mathcal{M}(F) & \to & \mathcal{N}(F) \\ (C,A) & \mapsto & A \end{cases}$$

While, already for simplicial complexes, a projection can create homology (see Figure 8), this can be controlled under certain assumptions. Let $\varphi : X \to Y$ be a monotone map between two simplicial posets. We say that φ is *dimension-preserving* if for any $\sigma \in X$ the dimension of $\varphi(\sigma)$ equals the dimension of σ; the *multiplicity* of φ is the maximum number of elements in the preimage of a simplex, that is, $\max_{\tau\in Y} |\varphi^{-1}(\tau)|$.

THEOREM 7. *If $\varphi : X \to Y$ is a surjective, dimension-preserving, monotone map of multiplicity r from a simplicial poset X to a simplicial complex Y, then $J(Y)+1 \le r(J(X)+1)$.*

We only outline the strategy of the proof. In the special case where X is a simplicial complex, the statement was proven by Kalai and Meshulam [33, Theorem 1.3]. (Kalai and Meshulam use *Leray numbers* instead of J-indices, but for simplicial complexes the two notions can be seen to coincide, from a characterization of Leray numbers in terms of the homology of links [32, Proposition 3.1]. Incidentally, we do not know whether these two notions coincide for simplicial posets.) Their proof uses a spectral sequence of Goryunov and Mond [25] that, given a map $f : X \to Y$ between two topological spaces, computes the homology of Y in terms of the homology of X provided f is "nice" enough. The main technical difficulty in extending that proof to maps between simplicial posets is that it is not clear whether there is a generalization of the notion of links to simplicial posets that leads to a similar characterization. We introduced the index J precisely as a way around that technical difficulty, and the rest of the proof extends with minor technical modifications.

We can finally conclude the proof of the announced bound on Helly numbers.

PROOF OF THEOREM 4. Let F be a finite acyclic family of open subsets of Γ, and assume that any subfamily of F intersects in at most r connected components. Let $\mathcal{N}(F)$ and $\mathcal{M}(F)$ denote, respectively, the nerve and the multinerve of F. We consider the projection:

$$\pi : \begin{cases} \mathcal{M}(F) & \to & \mathcal{N}(F) \\ (C,A) & \mapsto & A. \end{cases}$$

Each simplex in the pre-image $\pi^{-1}(\sigma)$ of a simplex $\sigma \in \mathcal{N}(F)$ is of the form (C,σ) where C is a connected component of $\bigcap_\sigma$. The projection π is therefore at most r-to-one and we can apply Theorem 7 with $X = \mathcal{M}(F)$ and $Y = \mathcal{N}(F)$. We obtain that $J(\mathcal{N}(F)) \le rJ(\mathcal{M}(F)) + r - 1$. With Lemma 6, this becomes $J(\mathcal{N}(F)) \le r(d_\Gamma + 1) - 1$. With Lemma 5, we get that the Helly number of F is at most $J(\mathcal{N}(F)) + 1$, which concludes the proof. □

4.3 Extension of Theorem 4 and Relation to Previous Work

The conclusion of Theorem 4 does, in fact, hold even if the intersection of small subfamilies has more than r connected components and is acyclic with some slack. Specifically, we prove the following strengthening of Theorem 4:

THEOREM 8. *Let F be a finite family of open subsets of a locally arc-wise connected topological space Γ. If (i) F is acyclic with slack s and (ii) any subfamily of F of cardinality at least t intersects in at most r connected components, then the Helly number of F is at most $r(\max(d_\Gamma, s, t) + 1)$.*

Due to space limitations, we omit the proof. In both Theorems 4 and 8 the openness condition can be replaced by a compactness condition under an additional mild assumption (for instance that the elements in F are subcomplexes of a given triangulation).

Finally, we remark that Theorem 8 differs from Matoušek's topological theorem on two accounts. First, his proof only gives a loose bound on the Helly number (in fact, no explicit bound is given), whereas our approach gives sharp, explicit, bounds. Second, his theorem allows the connected components to have nontrivial *homotopy* in *high* dimension, whereas Theorem 8 lets them have nontrivial *homology* in *low* dimension.

5. APPLICATION TO HELLY NUMBERS IN GEOMETRIC TRANSVERSAL THEORY

Let $\mathcal{H} = \{A_1, \ldots, A_n\}$ be a family of pairwise disjoint convex sets in $\mathbb{R}^d$ and let $T(\mathcal{H})$ denote the set of *line transversal* to $\mathcal{H}$, that is of lines[5] intersecting every member in $\mathcal{H}$. An important problem in geometric transversal theory is to understand which geometric conditions on the A_i allow to

[5]All lines are considered non-oriented.

bound the Helly number of $\{T(A_1), \ldots, T(A_n)\}$ (we refer to the classical surveys [16, 19, 31, 44] for the history of that problem). In this section we show that Theorem 8 gives, in a single stroke, three bounds on Helly numbers on sets of lines:

(i) Santaló's theorem [39] asserts that when the A_i form a finite family of parallelotopes in $\mathbb{R}^d$ with edges parallel to the coordinate axis, the Helly number of $\{T(A_i)\}_i$ is at most $2^{d-1}(2d-1)$. The same bound follows from Theorem 8.

(ii) Tverberg's transversal theorem [43] states that when the A_i are disjoint translates of a compact convex subset $D \subset \mathbb{R}^2$ with non-empty interior, the Helly number of $\{T(A_i)\}_i$ is at most 5. A bound of 10 follows from Theorem 8.

(iii) Cheong et al. [13] showed that if the A_i are disjoint unit balls in $\mathbb{R}^d$, then the Helly number of $\{T(A_i)\}_i$ is at most $4d-1$. Theorem 8 implies a sharper bound of $4d-2$ for $d \geq 6$ (and weaker bounds for $2 \leq d \leq 5$).

That (some weaker form of) these three results can be traced back to the same principle is new and interesting since so far, only ad hoc proofs were known. Also note that the theorems of Tverberg and Cheong et al. were conjectured in the 1950's by Grünbaum and Danzer, respectively, but only proven in 1989 and 2006, respectively. Having a tool, such as Theorem 8, that allows to identify situations where the Helly numbers of sets of transversals are bounded is therefore interesting in its own right.

Due to space limitations, we give complete details for example (iii) and refer to the full version for examples (i) and (ii).

5.1 Ambient Space

The space of lines in $\mathbb{R}^d$ can be considered as a subspace of the space of lines in $\mathbb{RP}^d$, which is the Grassmannian $\mathbb{RG}_{2,d+1}$ of all 2-planes through the origin in $\mathbb{R}^{d+1}$; the space $\mathbb{RG}_{2,d+1}$ is a compact manifold of dimension $2d-2$ and can be seen as an algebraic sub-variety of some $\mathbb{RP}^m$ via Grassmann coordinates (also known as Plücker coordinates for $d=3$). We note that $d_{\mathbb{RG}_{2,d+1}} \leq 2d-1$ [26, p. 121]. However, in the applications below, we consider the set Γ of lines in $\mathbb{R}^d$, which is a non-compact submanifold of dimension $2d-2$ of $\mathbb{RG}_{2,d+1}$. It follows that $d_\Gamma \leq 2d-2$ (again by [26, p. 121]).

5.2 Number and Homology of Connected Components

For any $A_i \in \mathcal{H}$ the set $T_i = T(\{A_i\})$ has the homotopy type of $\mathbb{RP}^{d-1}$, and is therefore homologically trivial in dimension d and higher. Let $G \subseteq \mathcal{H}$ be a subset of size 2 or more. To discuss the number and homology of the connected components of $T(G)$ we first consider the map $p : \mathbb{RG}_{2,d+1} \to \mathbb{RP}^{d-1}$ associating each line to its direction. Since the restriction $p_{|T(\mathcal{H})}$ is a closed map with contractible fibers, it follows from the Vietoris-Begle mapping theorem [41, Theorem 15, Section 6.9] that $T(G)$ and $p(T(G))$ have the same homology. The *geometric permutation* on F induced by a line transversal to a family F of disjoint convex sets is the pair of orderings, one reverse of the other, induced on F by the two orientations of the line. Since G consists of disjoint balls, the connected components of $p(T(G))$ are convex[6] and in one-to-one correspondence with the geometric permutations of G [7]. It follows that the connected components of $T(G)$ have the same homology as convex sets, that is are homology cells. Altogether, we get that $\mathcal{H}$ is acyclic with slack d. Moreover, for $G \subseteq \mathcal{H}$ the number of connected components of $T(G)$ is the same as the number of geometric permutations of G, which is at most 3 in general and at most 2 when $n \geq 9$ [14] (independently of the dimension).

5.3 Wrapping Up

We can now apply Theorem 8 with $d_\Gamma = 2d-2$, $s = d+1$, $t = 9$, and $r = 2$, obtaining the upper bound of $2\max(2d-1, 10)$. For $d \geq 6$, this yields the upper bound of $4d-2$, but for $d \in \{2,3,4,5\}$ this bound is only 20. In the case $d=2$ (resp. $d=3$) it can be improved to 12 (resp. 15) by using $d_\Gamma = 2d-2$, $s = d+1$, $t = 1$, and $r = 3$. Note that it is conjectured that any family of 4 or more disjoint equal-radius balls in $\mathbb{R}^d$ has at most two geometric permutations. If this is true, then our bounds would improve to $4d-2$ for any $d \geq 3$.

When $\mathcal{H} = \{A_1, \ldots, A_n\}$ is a collection of disjoint unit balls in $\mathbb{R}^d$, the Helly number of $\{T(A_i)\}_i$ is at least $2d-1$ [12], so this number is known up to a factor of 2. If we allow the balls to have arbitrary radii then the maximum number of geometric permutations becomes $\Theta(n^{d-1})$ [40] and the Helly number of $\{T(A_i)\}_i$ is unbounded. If the radii are required to be in some fixed interval $[1, \rho]$, the upper bound on the number of geometric permutations reduces to $O(\rho^{\log \rho})$ [45] and Theorem 8 similarly implies that the Helly number of $\{T(A_i)\}_i$ is $O(d\rho^{\log \rho})$, where the constant in the $O()$ is independent of ρ, n and d.

6. REFERENCES

[1] N. Alon and G. Kalai. Bounding the piercing number. *Discrete & Computational Geometry*, 13:245–256, 1995.

[2] N. Amenta. Helly-type theorems and generalized linear programming. *Discrete & Computational Geometry*, 12:241–261, 1994.

[3] N. Amenta. A new proof of an interesting Helly-type theorem. *Discrete & Computational Geometry*, 15:423–427, 1996.

[4] D. Attali and H. Edelsbrunner. Inclusion-exclusion formulas from independent complexes. *Discrete & Computational Geometry*, 37:59–77, 2007.

[5] D. Attali, A. Lieutier, and D. Salinas. Efficient data structure for representing and simplifying simplicial complexes in high dimensions. In *Proceedings of the 27th Annual ACM Symposium on Computational Geometry*, SoCG '11, pages 501–509, 2011.

[6] D. Attali, A. Lieutier, and D. Salinas. Vietoris-Rips complexes also provide topologically correct reconstructions of sampled shapes. In *Proceedings of the 27th Annual ACM Symposium on Computational Geometry*, SoCG '11, pages 491–500, 2011.

[7] C. Borcea, X. Goaoc, and S. Petitjean. Line transversals to disjoint balls. *Discrete & Computational Geometry*, 1-3:158–173, 2008.

[6] Convexity in $\mathbb{RP}^{d-1}$ is relative to the metric induced through the identification $\mathbb{RP}^{d-1} = \mathbb{S}^{d-1}/\mathbb{Z}_2$.

[8] K. Borsuk. On the imbedding of systems of compacta in simplicial complexes. *Fundamenta Mathematicae*, 35:217–234, 1948.

[9] R. Bott and L. W. Tu. *Differential forms in algebraic topology*, volume 82 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1982.

[10] G. E. Bredon. *Sheaf theory*, volume 170 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1997.

[11] F. Chazal and S. Y. Oudot. Towards persistence-based reconstruction in Euclidean spaces. In *Proceedings of the 24th Annual Symposium on Computational Geometry*, SoCG '08, pages 232–241, 2008.

[12] O. Cheong, X. Goaoc, and A. Holmsen. Lower bounds to Helly numbers of line transversals to disjoint congruent balls. *Israel Journal of Mathematics*, 2012. To appear.

[13] O. Cheong, X. Goaoc, A. Holmsen, and S. Petitjean. Hadwiger and Helly-type theorems for disjoint unit spheres. *Discrete & Computational Geometry*, 1-3:194–212, 2008.

[14] O. Cheong, X. Goaoc, and H.-S. Na. Geometric permutations of disjoint unit spheres. *Computational Geometry: Theory & Applications*, 30:253–270, 2005.

[15] É. Colin de Verdière, G. Ginot, and X. Goaoc. Helly numbers of acyclic families. arXiv:1101.6006, 2011.

[16] L. Danzer, B. Grünbaum, and V. Klee. Helly's theorem and its relatives. In V. Klee, editor, *Convexity*, Proceedings of Symposia in Pure Mathematics, pages 101–180. AMS, 1963.

[17] V. de Silva and G. Carlsson. Topological estimation using witness complexes. *IEEE Symposium on Point-Based Graphics*, pages 157–166, 2004.

[18] H. Debrunner. Helly type theorems derived from basic singular homology. *American Mathematical Monthly*, 77:375–380, 1970.

[19] J. Eckhoff. Helly, Radon and Caratheodory type theorems. In J. E. Goodman and J. O'Rourke, editors, *Handbook of Convex Geometry*, pages 389–448. North Holland, 1993.

[20] J. Eckhoff and K.-P. Nischke. Morris's pigeonhole principle and the Helly theorem for unions of convex sets. *Bulletin of the London Mathematical Society*, 41:577–588, 2009.

[21] H. Edelsbrunner and N. R. Shah. Triangulating topological spaces. *International Journal of Computational Geometry and Applications*, 7(4):365–378, 1997.

[22] B. Farb. Group actions and Helly's theorem. *Advances in Mathematics*, 222:1574–1588, 2009.

[23] R. Godement. *Topologie algébrique et théorie des faisceaux*. Hermann, Paris, 1973. Troisième édition revue et corrigée, Publications de l'Institut de Mathématique de l'Université de Strasbourg, XIII, Actualités Scientifiques et Industrielles, No. 1252.

[24] P. G. Goerss and J. F. Jardine. *Simplicial homotopy theory*, volume 174 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 1999.

[25] V. Goryunov and D. Mond. Vanishing cohomology of singularities of mappings. *Compositio Mathematica*, 89(1):45–80, 1993.

[26] M. J. Greenberg. *Lectures on Algebraic Topology*. Benjamin, Reading, MA, 1967.

[27] B. Grünbaum and T. Motzkin. On components in some families of sets. *Proceedings of the American Mathematical Society*, 12:607–613, 1961.

[28] A. Hatcher. *Algebraic topology*. Cambridge University Press, 2002. Available at `http://www.math.cornell.edu/~hatcher/`.

[29] E. Helly. Über Mengen konvexer Körper mit gemeinschaftlichen Punkten. *Jahresbericht der Deutschen Mathematiker-Vereinigung*, 32:175–176, 1923.

[30] E. Helly. Über Systeme von abgeschlossenen Mengen mit gemeinschaftlichen Punkten. *Monatshefte für Mathematik und Physik*, 37:281–302, 1930.

[31] A. Holmsen. Recent progress on line transversals to families of translated ovals. In J. E. Goodman, J. Pach, and R. Pollack, editors, *Computational Geometry - Twenty Years Later*, pages 283–298. AMS, 2008.

[32] G. Kalai and R. Meshulam. Intersections of Leray complexes and regularity of monomial ideals. *Journal of Combinatorial Theory, Series A*, 113:1586–1592, 2006.

[33] G. Kalai and R. Meshulam. Leray numbers of projections and a topological Helly-type theorem. *Journal of Topology*, 1:551–556, 2008.

[34] M. Kashiwara and P. Schapira. *Sheaves on manifolds*, volume 292 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1990.

[35] J. Matoušek. *Using the Borsuk-Ulam Theorem*. Springer-Verlag, 2003.

[36] J. Matoušek. A Helly-type theorem for unions of convex sets. *Discrete & Computational Geometry*, 18:1–12, 1997.

[37] J. P. May. *Simplicial objects in algebraic topology*. Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 1992. Reprint of the 1967 original.

[38] H. Morris. *Two pigeon hole principles and unions of convexly disjoint sets*. Phd thesis, California Institute of Technology, 1973.

[39] L. Santaló. Un theorema sobre conjuntos de paralelepipedos de aristas paralelas. *Publ. Inst. Mat. Univ. Nat. Litoral*, 2:49–60, 1940.

[40] S. Smorodinsky, J. S. B. Mitchell, and M. Sharir. Sharp bounds on geometric permutations for pairwise disjoint balls in $\mathbb{R}^d$. *Discrete & Computational Geometry*, 23:247–259, 2000.

[41] E. H. Spanier. *Algebraic topology*. McGraw-Hill Book Co., New York, 1966.

[42] R. Stanley. f-vectors and h-vectors of simplicial posets. *J. Pure and Applied Algebra*, 71:319–331, 1991.

[43] H. Tverberg. Proof of Grünbaum's conjecture on common transversals for translates. *Discrete & Computational Geometry*, 4:191–203, 1989.

[44] R. Wenger. Helly-type theorems and geometric transversals. In J. E. Goodman and J. O'Rourke, editors, *Handbook of Discrete & Computational*

Geometry, chapter 4, pages 73–96. CRC Press LLC, Boca Raton, FL, 2nd edition, 2004.

[45] Y. Zhou and S. Suri. Geometric permutations of balls with bounded size disparity. *Computational Geometry: Theory & Applications*, 26:3–20, 2003.